Q.34 If $\vec{A} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$, $\vec{B} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$.

Find (i) $\frac{d}{dt}(\vec{A} \times \vec{B})$ at $t=1$

(ii) $\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{B}}{dt}\right)$ at $t=1$.

(i) $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= (t^2 - 2t - 1)\hat{i} - [-t^3 - (4t^2 - 4t - 3)]\hat{j} + (t^2 + 2t^2 - 3t)\hat{k}$$

$$= (t^2 - 2t - 1)\hat{i} + (t^3 + 4t^2 - 4t - 3)\hat{j} + (3t^2 - 3t)\hat{k}$$

$$\therefore \frac{d}{dt}(\vec{A} \times \vec{B}) = (2t - 2)\hat{i} + (3t^2 + 8t - 4)\hat{j} + (6t - 3)\hat{k}$$

At $t=1$; $\frac{d}{dt}(\vec{A} \times \vec{B}) = 0 + (11 - 4)\hat{j} + (6 - 3)\hat{k}$
$= 7\hat{j} + 3\hat{k}$ Ans.

Q.12. P-42 A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$, where ω is constant. Show that
(i) the velocity $\vec{v}$ of the particle is perp. to $\vec{r}$.
(ii) the acc^n $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.

(i) $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

$\vec{v} = \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$

Now $\vec{r}\cdot\vec{v} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})\cdot(-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})$

$= -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t$

$= 0$ $\quad\therefore \vec{v}$ is perp. to $\vec{r}$. proved

(ii) Acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

$$= \frac{d}{dt}\left(-\omega \sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}\right)$$
$$= -\omega^2 \cos\omega t\,\hat{i} - \omega^2 \sin\omega t\,\hat{j}$$
$$= -\omega^2\left(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right)$$
$$= -\omega^2 \vec{r}$$

$\therefore$ Acceleration is opposite to the direction of $\vec{r}$ i.e it is directed towards the origin and its magnitude is proportional to $\vec{r}$ which is the distance from the origin.

$$\vec{a} = -\omega^2\vec{r}$$
$$|\vec{a}| = |-\omega^2\vec{r}|$$
$$|\vec{a}| = |\omega^2\vec{r}|$$
$$|\vec{a}| \propto |\vec{r}|$$

Q44 / P-54. If $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$; $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$

find $\frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B})$ at the point $(1, 0, -2)$.

Ans:- $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$; $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$.

$$\therefore \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} - (-x^4yz - 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} + (x^4yz + 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}$$

Now $\frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{(2x^3z^3 - xyz^2)\,\hat{i} + (x^4yz + 2xz^3)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}\right\}\right]$

$$= \frac{\partial}{\partial x}\left[-xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}\right)$$
$$= -z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}.$$

$\therefore$ At the point $(1,0,-2)$; $\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = -4\hat{i} - 8\hat{j}$. Ans.

Q.45 / P-54. If $\vec{e_1}$ and $\vec{e_2}$ are constant vectors and λ is a constant scalar, show that $\vec{H} = e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)$ satisfies the partial differential equation $\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = 0$.

Ans:- Given $\vec{H} = e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)$, then

LHS $\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} = \frac{\partial}{\partial x}\left[-\lambda e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)\right]$

$+ \frac{\partial}{\partial y}\left[e^{-\lambda x}(\lambda\vec{e_1}\cos\lambda y - \lambda\vec{e_2}\sin\lambda y)\right]$

$= \lambda^2 e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)$

$+ e^{-\lambda x}(-\lambda^2\vec{e_1}\sin\lambda y - \lambda^2\vec{e_2}\cos\lambda y)$

$= \lambda^2 e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)$

$- \lambda^2 e^{-\lambda x}(\vec{e_1}\sin\lambda y + \vec{e_2}\cos\lambda y)$

$= \lambda^2\vec{H} - \lambda^2\vec{H}$

$= 0$. Proved.

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